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CORRECTION

COMPLETE CLASSES FOR SEQUENTIAL TESTS OF HYPOTHESES

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Theorems 3.1 and 3.2 as stated are incorrect. Corrected versions of these results are given below. Theorem 3.1 was concerned with an essentially complete class. Theorem 3.2 was concerned with a complete class. The corrections do affect a qualitative change in Theorem 3.2 in that now the result requires an assumption of a one dimensional exponential family and treats only a one-sided testing problem. There is essentially no qualitative change in Theorem 3.1, where the assumptions on distributions are minimal and there are no changes in the rest of the paper.

The new version of Theorem 3.1 is also concerned with an essentially complete class. To describe this class let $\mathscr{D}^*$ be the class of procedures characterized by $(\gamma, \rho, \pi_1^*, \Gamma_1^*, \Gamma_2^*)$ where γ is the probability of stopping at time zero, ρ is the probability of rejection given that the procedure stopped at time zero and $(\pi_1^*, \Gamma_1^*, \Gamma_2^*)$ are defined on pages 384 and 385, $\pi_1^* \geq c$. A procedure δ corresponding to a $(\gamma, \rho, \pi_1^*, \Gamma_1^*, \Gamma_2^*)$ lies in $\mathscr{D}^*$ if whenever $0 \leq \gamma < 1$, δ is conditional Bayes with respect to $(\pi_1^*, \Gamma_1^*, \Gamma_2^*)$ given that an observation has been taken.

THEOREM 3.1′. *The class $\mathscr{D}^*$ is essentially complete.*

PROOF. The error made in the paper involves Lemma 3.1 and its subsequent use, *only when* $n = 0$. Note that for $n = 0$, $g^{(n)}_{(i),k}(x_{(0)}) \equiv 1$ since the σ-field generated by $X_{(0)}$ is trivial.

Thus $\pi_{ik'} g^{(0)}_{(i),k'}(x_{(0)}) \equiv \pi_{ik'} \to \tilde{\pi}_i$ [instead of $\pi_{ik'} g^{(0)}_{(i),k'}(x_{(0)}) \to \pi_i^* g^{(0)}_{(i),*}(x_{(0)})$ as claimed in Lemma 3.1]. Corresponding minor modifications need to be made for the case $n = 0$ in Lemma 3.2 and in the proof of Theorem 3.1 on pages 386 and 387. The outcome of these modifications is that the limiting rule $\delta(\overline{X})$ is Bayes with respect to $(\pi_1^*, \Gamma_1^*, \Gamma_2^*)$ only after $X_{(1)}$ has been observed. Before $X_{(1)}$ is observed (i.e., for $n = 0$) δ will stop and accept (reject) if $\tilde{\pi}_1 > \frac{1}{2}$ $(\tilde{\pi}_2 > \frac{1}{2})$ and $\min(\tilde{\pi}_1, \tilde{\pi}_2) < \beta_{n,*}(x_{(0)})$, whereas the Bayes procedure for $(\pi_1^*, \Gamma_1^*, \Gamma_2^*)$ substitutes π_i^* for $\tilde{\pi}_i$ in the preceding rules. Since $\tilde{\pi}_1 \leq \pi_1^*$ and $\tilde{\pi}_2 \geq \pi_2^*$ it follows that δ can reject at stage 0 when the Bayes procedure does not or δ (based on $\tilde{\pi}_1$) would say to continue but the Bayes procedure (based on π_1^*) would say to stop and accept. Hence δ is Bayes only after an observation has been taken. □

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REMARK. It is possible to construct a procedure in $\mathscr{D}^*$ which is admissible but not Bayes. This shows that the original Theorem 3.1 is false.

Theorem 3.2 can be replaced by a theorem requiring additional assumptions. We require that $X_1, X_2, \ldots$ are independent, identically distributed random variables from a continuous parameter exponential family. The density is in terms of the natural statistic x and natural parameter θ. We test $\theta \in \overline{\Theta}_1$ versus $\theta \in \overline{\Theta}_2$ where θ's in $\overline{\Theta}_2$ are greater than or equal to all θ's in $\overline{\Theta}_1$, that is, we test a one-sided alternative.

THEOREM 3.2′. *$\mathscr{D}^*$ is a complete class.*

PROOF. Suppose $\delta' \notin \mathscr{D}^*$ and assume δ' is admissible. Otherwise it is clear from Theorem 3.1′ that there exists a test in $\mathscr{D}^*$ which is better. Let $\delta \in \mathscr{D}^*$ be such that $R(\theta, \delta) = R(\theta, \delta')$ for every real θ. Then

$$\begin{aligned} &\gamma(1-p) + (1-\gamma)c + (1-\gamma)R_1(\theta,\delta) \\ &\quad = \gamma'(1-p') + (1-\gamma')c + (1-\gamma')R_1(\theta,\delta') \quad \text{for } \theta \in \overline{\Theta}_2 \end{aligned} \tag{1}$$

and

$$\gamma p + (1-\gamma)c + (1-\gamma)\tilde{R}_1(\theta,\delta) = \gamma' p' + (1-\gamma')c + (1-\gamma')\tilde{R}_1(\theta,\delta') \quad \text{for } \theta \in \overline{\Theta}_1, \tag{2}$$

where $R_1(\theta,\delta)$ is the risk of δ for $\theta \in \overline{\Theta}_2$ given that the first observation is free and taken and $\tilde{R}_1(\theta,\delta)$ is defined similarly for $\theta \in \overline{\Theta}_1$.

Since $\delta' \notin \mathscr{D}^*$, $\gamma' \neq 1$, otherwise $\delta' \in \mathscr{D}^*$. Also δ' does not accept at stage 1 with probability 1, otherwise it would lie in $\mathscr{D}^*$. This, plus the fact that δ' is admissible implies that $R_1(\theta,\delta') \to 0$ as $\theta \to \infty$. This follows from the facts that (i) among procedures which observe at least once, $\overline{\mathscr{B}}$ is a complete class [see Brown, Cohen and Strawderman (1980), page 395, Theorem 4.2] and (ii) every procedure in $\overline{\mathscr{B}}$ is monotone [see Brown, Cohen and Strawderman (1979), page 1228, Remark 3.4] and furthermore every such procedure (except one which always accepts at stage 1) must reject for x_1 sufficiently large. Similarly δ does not accept at stage 1 with probability 1 since the procedure which accepts at stage 0 with probability $(1-\gamma p)$ and rejects at stage 0 with probability γp is better. It follows that $R_1(\theta,\delta) \to 0$ as $\theta \to \infty$ as well. [Note: If $\gamma = 1$, take $R_1(\theta,\delta) \equiv 0$.]

The properties above for δ and δ' along with (1) imply

$$R_1(\theta,\delta) = BR_1(\theta,\delta') \quad \text{for all } \theta \in \overline{\Theta}_2, \tag{3}$$

where $B = (1-\gamma')/(1-\gamma)$, $0 \le \gamma < 1$, $0 \le \gamma' < 1$. (The case $\gamma = 1$ implies $\gamma' = 1$, which was ruled out.) Hence $0 < B < \infty$. Furthermore $B \neq 1$. To see this suppose $B = 1$, i.e., $\gamma = \gamma'$. Then from (3) and (1), $p = p'$, which in turn implies $\tilde{R}_1(\theta,\delta) = \tilde{R}_1(\theta,\delta')$, which in turn with (3) implies $\delta' \in \mathscr{D}^*$.

We will show that $[R_1(\theta, \delta)/R_1(\theta, \delta')]$ is not constant for all $\theta \in \overline{\Theta}_2$. This contradicts (3). Consider

$$R_1(\theta, \delta) = P_\theta(A_\delta;\, N_\delta = 1) + cP_\theta(N_\delta > 1) + R_2(\theta, \delta)P_\theta(N_\delta > 1), \tag{4}$$

where A_δ is the acceptance set for δ, $R_2(\theta, \delta)$ is the risk for given δ given at least two observations are taken and $\theta \in \overline{\Theta}_2$, N_δ is the stopping time for δ. Clearly $R_2(\theta, \delta) \to 0$ as $\theta \to \infty$ as does $R_2(\theta, \delta')$ by the same reasoning used before for $R_1(\theta, \delta)$. Now for δ at stage 1, there are the following two possibilities.

(i) Stop and accept if $x_1 < a_1$, continue if $a_1 < x_1 < b_1$, stop and reject if $x_1 > b_1$.

(ii) Stop and accept if $x_1 < a_1$, stop and reject if $x_1 > a_1$. Similarly for δ' we have corresponding a_1' and b_1'.

When δ and δ' are of form 1 with $b_1 < b_1'$ we have

$$\begin{aligned}\frac{R_1(\theta, \delta)}{R_1(\theta, \delta')} &= \frac{P_\theta(N_\delta > 1)\big(c + R_2(\theta, \delta) + [P_\theta(A_\delta;\, N_\delta = 1)]/[P_\theta(N_\delta > 1)]\big)}{P_\theta(N_{\delta'} > 1)\big(c + R_2(\theta, \delta') + [P_\theta(A_{\delta'};\, N_{\delta'} = 1)]/[P_\theta(N_{\delta'} > 1)]\big)} \\ &\le \frac{P_\theta(N_\delta > 1)}{P_\theta(N_{\delta'} > 1)}\left(\left(c + (1 + c) + \frac{P_\theta(A_\delta;\, N_\delta = 1)}{P_\theta(N_\delta > 1)}\right)\Big/ c\right) \to 0\end{aligned} \tag{5}$$

as $\theta \to \infty$ by using the fact that X_1 has an exponential family distribution. Clearly when the ratio on the left-hand side of (5) tends to 0 or ∞ as $\theta \to \infty$, the ratio is not constant. Similarly $b_1 > b_1'$ is impossible, so $b_1 = b_1'$. If $b_1 = b_1'$, then the ratio in (5) tends to 1 as $\theta \to \infty$, which contradicts the fact that $B = 1$ is excluded.

If both δ and δ' are of form 2, then $a_1 = a_1'$ and again $B = 1$. Finally suppose one of δ, δ' satisfies form 1 (say δ) and the other (say δ') satisfies form 2. Then for $\theta \in \overline{\Theta}_2$ the ratio on the left-hand side of (5) is

$$c + (1 - c)\frac{F_\theta(a_1)}{F_\theta(b_1)} + \frac{R_2(\theta, \delta)(F_\theta(b_1) - F_\theta(a_1))}{F_\theta(b_1)}, \tag{6}$$

where $F_\theta(\cdot)$ is the c.d.f. of X. Note (6) is not constant for $\theta \in \overline{\Theta}_2$. $\square$

Remark. Theorem 3.2′ can be extended to the two-sided symmetric case.

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